PUBLIC GOODS IN NETWORKS: SOME RESULTS

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ABSTRACT. This paper introduces a new mechanism through which positive externality spills over in a network. We find that the introduction of the new spillover mechanism plays a major role in reducing the set of Nash equilibrium though at the same time it increases the set of stable equilibria in a network game with local positive externality. Furthermore, we show that a denser network do not necessarily give rise to higher welfare.

Keywords: Innovation, Network, Maximal independent set, Nash tatonnement, Welfare

JEL Classification: D83, D85, H41, O31

1. Introduction

This paper investigates how prevailing networks in a society can affect individual incentives to generate new information or invest in R&D that creates local positive externality; that is once generated it becomes non-excludable along direct social links. Typically, innovations are gradual and technology progresses incrementally. A firm has to exert effort for bringing about innovation, with higher effort yielding better progress although at a greater cost, but once it happens it assumes the nature of a local public good. Hence with various firms innovating simultaneously, once a firm exerts high effort the lower efforts by its peers are rendered unproductive. Thus it is the maximum of the efforts exerted by a firm and its peers that remains relevant in determining the state of the technology available in an industry at a certain point of time. This idea motivates a model of public good provision where the level of public good depends on the maximum of contributions by the agents in a neighbourhood[1]. To this end, we model a typical agent with a payoff function that allows for dependence of her benefit on the contribution of other agents; the structure of the benefit reflecting zero marginal return from one's contribution if it is

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[1]The importance of such setting was first pointed out in the context of standard problem of voluntary provision of public good in Hirshleifer (1983), Harrison and Hirshleifer (1989).

lower than someone else's contribution in her neighbourhood.

This paper situates itself in a growing body of literature on games in networks, an exhaustive survey of which is provided in Jackson (2008) and Goyal (2007), Jackson and Zenou (2014). An important strand of the literature to which this paper contributes analyses how a given network shapes the equilibrium outcomes of a game. While Bramoulle and Kranton (2007) provides such an analysis for undirected networks in the context of public good provision, López-Pintado (2013) extends it for directed networks. Bramoulle et al. (2014) further explores how network link patterns affect the equilibrium through strategic interaction for a broader class of games. Most of the literature however focuses its attention on games where actions of different agents are perfect or imperfect substitutes of each other with the exception of Ballester et al. (2006) and Corbo et al. (2007) which allows complementarity in the strategies of different agents in their analysis. While such settings find many useful applications, there is still a wide variety of other settings exploring which may be worthwhile.

Our model, aimed at analysing the setting discussed above, is a simple adaptation of the model of public goods in Bramoulle and Kranton (2007) but our point of departure is the way we incorporate the spillover of the benefits of contribution within a network. We propose a non-cooperative simultaneous-move game in which given the network structure within which an individual is embedded she decides whether and how much effort to put towards public good provision when the maximum of the efforts of her own self and her neighbours determine the level of public good she has access to[2]. Our model predicts the possible equilibria of the above game and examines their stability. We observe that this slight modification of the model of Bramoulle and Kranton (2007) introduces substantive changes in the conclusions of their analysis.

While we focus on a particular example for our analysis our findings are relevant in contexts like information gathering by individuals wherein people often turn to those close ties for information who are known to have invested substantially in acquiring those information making the marginal contribution of others in her network zero.

[2]The Best Shot network game that has been analysed in the literature on games on literature (see Jackson (2008), Galeotti et al. (2010), Boncinelli and Pin (2012)) comes very close to our setting though in such games individuals only have a choice between exerting or not exerting effort.

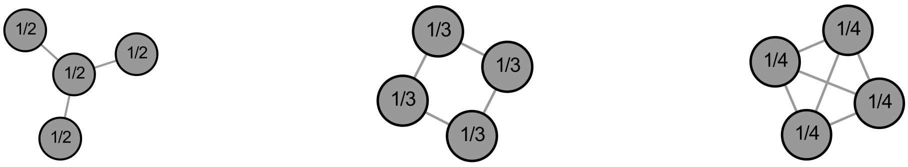
The main insights of our paper are: First, social networks give rise to only specialised equilibrium. Second, all such equilibria are stable. Thus the stability analysis does not take us much far in improving the predictive power of the model by selecting among the multiple equilibria possible under this set up. Third, new links can reduce social welfare. The rest of the paper is organised as follows. Section 2 presents the model, section 3 discusses the results, section 4 deals with some extensions of our basic model and section 5 concludes.

2. **Model**

Consider a set of agents N={1,..., n} who may be linked to each other in a network. A network of agents is represented by an undirected graph **g** which is a collection of links $g_{ij} \in \{0,1\}$ for all $i, j \in N$. Each agent represents a node in the graph. For any two agents i and j,

$$\begin{aligned} g_{ij} = g_{ji} &= 1 \quad \text{if } i \text{ and } j \text{ are linked} \\ &\ 0 \qquad \text{otherwise} \end{aligned}$$

We call an agent $j \in N$ an immediate neighbour of $i \in N$ if she is linked to agent i. A path is said to exist between two agents i and j in **g** if either $g_{ij} = 1$ or $\exists$ distinct $j_1, \ldots, j_m \in N - \{i, j\}$ such that $g_{ij_1} = g_{j_1 j_2} = \ldots = g_{j_m j} = 1$. Let the geodesic distance between i and j, defined as the length of (number of links in) the shortest path between them in **g** be denoted by $d(i, j; \mathbf{g})$. Throughout the paper distance will refer to geodesic distance. If there is no path between i and j, we set $d(i, j; \mathbf{g}) = \infty$. We define $N_i^l(\mathbf{g}) = \{j \in N | d(i, j; \mathbf{g}) = l\}$ as the set of agents who are at a distance l from agent i where $l \in \{1, \ldots, n-1\}$. Therefore, $N_i^1(\mathbf{g}) = \{j \in N - \{i\} | g_{ij} = 1\}$ denote the set of all immediate neighbours of individual i. Any agent who is at a distance more than 1 from i will be called a distant neighbour of i. Define $N_i(\mathbf{g}) = \bigcup_{l=1}^{n-1} N_i^l(\mathbf{g})$ as the set of all agents who have a path with i and call any such agent a neighbour of i. Thus the set $N_i(\mathbf{g}) = \bigcup_{l=1}^{n-1} N_i^l(\mathbf{g})$ consists of all immediate as well as distant neighbours of i. If $i', j' \in N$ are two agents not linked in **g**, then $\mathbf{g} + g_{i'j'}$ denote the network obtained by connecting i' and j'. An independent set I of a network **g** is the set of agents such that no two



(a) (b) (c)

FIGURE 1. Distributed Profiles

(a) (b)

FIGURE 2. Specialised profiles

agents who belong to I are linked in $\mathbf{g}$. In other words, $(\forall i, j \in I)(i \neq j \rightarrow g_{ij} = 0)$. An independent set is maximal when it is not a proper subset of any other independent set. Each agent of a network $\mathbf{g}$ simultaneously chooses how much effort to exert. For a network $\mathbf{g}$, let $e_{i|_{\mathbf{g}}} \in [0, \infty)$ be the level of effort of agent $i \in N$. The set of effort levels of player i is denoted by E_i. For a given $\mathbf{g}$, let $\mathbf{e}_{|_{\mathbf{g}}}$ denote an effort profile of all individuals $i \in N$ and $\mathbf{e}_{\sim i|_{\mathbf{g}}}$ denote a profile of efforts of all $j \in N - \{i\}$. We will however drop the subscripts from the above notations when we are dealing with a single network throughout in a particular context. For each individual $i \in N$ the cost per unit of effort is assumed to be constant at c.

We assume that the benefit that an agent i derives from a network depends on the maximum of the efforts exerted by herself and her neighbours; no agent i derives any benefit from another individual j if $g_{ij} = 0$. The benefit function of agent i is given by $b(e_{i_{max}}), b' > 0, b'' < 0, b(0) = 0, b'(0) > c$ where $e_{i_{max}} = max\{e_j | j \in N_i^1(\mathbf{g}) \cup \{i\}\}$. Therefore, the payoff of agent i from an effort profile $\mathbf{e}$ is given by $U_i(\mathbf{e}; \mathbf{g}) = b(e_{i_{max}}) - ce_i$. The objective of each agent is to choose an effort level to maximise her own payoff.

Let e^* denote the optimal level of effort of an isolated individual, that is an individual who is not connected to any other agent. Therefore, $e^* = \arg\max_{e_i \in E_i} b(e_i) - ce_i$. Let $BR_i(\mathbf{e}_{\sim i}; \mathbf{g})$ denote the best response function of agent i which, given a profile of efforts of other agents in the network $\mathbf{g}$, gives the effort level that maximises her payoff.

An effort profile $\mathbf{e}$ is said to be specialised iff $e_i \in \{0, e^*\}$, $\forall i \in N$. On the other hand it is said to be distributed when $0 < e_i < e^*$, $\forall i \in N$. An agent who exerts e^* level of effort is called a specialist. We denote the set of specialists in $\mathbf{e}$ by $S(\mathbf{e})$. Figures 1 and 2 illustrate various networks with n=4 and $e^* = 1$. The effort profiles shown in all three networks in Figure 1 are distributed profiles. Figure 2 on the other hand demonstrates two different specialised effort profiles in the same network.

An effort profile $\mathbf{e}$ is a Nash equilibrium[3] for a network $\mathbf{g}$ iff $(\forall i \in N)(\forall e_i' \in E_i)[U_i(e_i, \mathbf{e}_{\sim i}; \mathbf{g}) \geq U_i(e_i', \mathbf{e}_{\sim i}; \mathbf{g})]$. An effort profile $\mathbf{e}$ is a strict Nash equilibrium if the above inequality is strict.

An equilibrium $\mathbf{e}$ is stable[4] iff $\exists \epsilon > 0$ such that for any vector γ satisfying $\forall i, |\gamma_i| \leq \epsilon \wedge e_i + \gamma_i \geq 0)$ the sequence $\mathbf{e}^{(n)}$ defined by $\mathbf{e}^{(0)} = \mathbf{e} + \gamma$ and $\mathbf{e}^{(\mathbf{n+1})} = BR(\mathbf{e}^{(\mathbf{n})})$ converges to $\mathbf{e}$.

We define social welfare under an effort profile $\mathbf{e}$ for a network $\mathbf{g}$ as the sum of individual payoffs. Under any profile $\mathbf{e}$ social welfare is given by $W(\mathbf{e}; \mathbf{g}) = \sum_{i \in N} U_i(\mathbf{e}; \mathbf{g})$. A profile $\mathbf{e}$ is said to be socially efficient iff $W(\mathbf{e}; \mathbf{g}) \geq W(\mathbf{e}'; \mathbf{g})$ for all $\mathbf{e}'$.

An equilibrium profile $\mathbf{e}$ is said to be second best for a network $\mathbf{g}$ if $W(\mathbf{e}; \mathbf{g}) \geq W(\mathbf{e}'; \mathbf{g})$ for all equilibrium profiles $\mathbf{e}'$.

3. **Results**

Our first proposition characterises the set of Nash equilibrium for any network $\mathbf{g}$ using the properties of the network. It states that no distributed profile qualifies as an equilibrium in a network. It further points out that in an equilibrium no two specialists are linked to each other. This is intuitive because once an agent takes the e^* level of effort her neighbours can free ride by deriving the benefit from her effort and not incurring any cost themselves. In addition Proposition 1 also asserts that all such specialised profiles in which the specialists are not linked to each other turn out to be the equilibrium profiles of our game. The second proposition tells that all equilibria in a network are stable. That is, sufficiently small perturbations from an equilibrium effort profile will necessarily restore

[3]Throughout the paper we will discuss only pure strategy equilibrium.

[4]We consider discrete Nash tatonnement in our analysis to examine stability of equilibria. See Fudenberg and Tirole (1991) for a description of the process of adjustment and Bramoulle and Kranton (2007) for the formal definition.

the original profile through the interplay of several best responses.

To derive the results formally we begin by noting that the fact that the increasing and strictly concave payoff functions of the agents guarantee the existence of an effort level $\overline{e} \in (0, e^*)$ such that

$$\begin{aligned} b(e^*) - ce^* &> b(e) && \text{for} && e < \overline{e} \\ &< b(e) && \text{for} && e > \overline{e} \\ &= b(e) && \text{for} && e = \overline{e} \end{aligned} \tag{3.1}$$

Therefore the best response function of an agent i is given by

$$BR_i(\mathbf{e}_{\sim i}; \mathbf{g}) = \begin{cases} \{e^*\} & \text{if} \quad (\forall j \in N_i^1(\mathbf{g}))(e_j < \overline{e}) \\ \{0, e^*\} & \text{if} \quad (\forall j \in N_i^1(\mathbf{g}))(e_j \leq \overline{e}) \wedge (\exists j \in N_i^1(\mathbf{g}))(e_j = \overline{e}) \\ \{0\} & \text{if} \quad (\exists j \in N_i^1(\mathbf{g}))(e_j > \overline{e}) \end{cases} \tag{3.2}$$

We now state our first proposition:

Proposition 1. *An effort profile* $\mathbf{e}$ *is a Nash Equilibrium for* $\mathbf{g}$ *iff it is specialisedand* $S(\mathbf{e})$ *is a maximal independent set of the network* $\mathbf{g}$.

Proof. Consider a Nash equilibrium profile $\mathbf{e}$. Since at Nash equilibrium all agents exert their best response effort levels, that $\mathbf{e}$ is specialised is immediate from 3.2 as every agent either exerts e^* level of effort or no effort at the equilibrium. Next we show that $S(\mathbf{e})$ is a maximal independent set. Consider an agent $i \in S(\mathbf{e})$. Therefore $(\forall j \in N_i^1(\mathbf{g}))(e_j = 0)$, i.e. none of the neighbours of i will exert any effort and hence cannot be specialists. This holds true for all specialists. Hence $S(\mathbf{e})$ must be an independent set of the network $\mathbf{g}$. Now consider any $k \in N - S(\mathbf{e})$. There must exist some $j \in N_k$ such that $e_j = e^*$. This means that all non-specialists are connected to at least one specialist. Hence $S(\mathbf{e})$ is a maximal independent set of the network $\mathbf{g}$. This proves the necessity part.

For the proof of sufficiency part, consider a specialised effort profile $\mathbf{e}$ such that $S(\mathbf{e}) \neq \phi$ and $S(\mathbf{e})$ is a maximal independent set of the network $\mathbf{g}$. Consider any specialist $i \in S(\mathbf{e})$.

$S(\mathbf{e})$ being a maximal independent set i is linked only to the non-specialists and hence gets the maximum payoff ($= b(e^*) - ce^*$); by exerting any other level of effort she derives lower payoff from the network. Now consider any non-specialist. She is linked to at least one specialist and acquires the benefit of $b(e^*)$. Therefore, by 3.2 she is playing her best response and gets no additional benefit by increasing her effort level. Since all the agents play their best responses, $\mathbf{e}$ is a Nash equilibrium. □

The existence of a Nash equilibrium is guaranteed by the fact that for every $\mathbf{g}$ there exists a maximal independent set. [5].

Example 1. *Nash equilibria and Maximal Independent sets: Consider a connected network[6] with four agents as illustrated in Figure 2 and Figure 1(b). Assume $e^* = 1$. For any network, only specialised profiles can be equilibrium. Therefore, the effort profile shown in Figure 1(b) is not an equilibrium. Moreover, in this network since every agent has exactly two immediate neighbours, in all equilibrium profiles on this network every alternate agent must be a specialist while the other agents completely free ride, an example of which is shown in Figure 2(a). In contrast, Figure 2(b) shows a specialised profile which is not an equilibrium as two specialists are linked to each other and hence the set of specialists is not a maximal independent set of the graph. There can be at most two such equilibria as there are two maximal independent sets where alternate agents are specialists.*

3.1. **Stability.** The standard technique to refine the equilibrium set is to look for stable equilibria. However, the selecting power of Nash tatonnement is rather limited in this model as is demonstrated in the main result of this section which shows that all Nash equilibria are stable. The argument is straightforward. Consider any equilibrium of a network $\mathbf{g}$. Any non-specialist i must be linked to at least one specialist $j \in S(\mathbf{e})$. If the effort level of the agent i is increased marginally, agent j will not adjust its effort as her payoff decreases. Similarly, if the effort level of a specialist $j \in S(\mathbf{e})$ is reduced by a very small amount agent i will not adjust its efforts as the additional benefit she will derive from such adjustment will be much less than the cost incurred. Our next proposition formally proves the result.

[5]See Bramoulle and Kranton (2007) for the existence of maximal independent set of a graph.
[6]A network is connected if between every two nodes there is a path which connects them.

(a) (b)

FIGURE 3. Star

Proposition 2. *For any network* $\mathbf{g}$*, all equilibria are stable.*

Proof. Consider a Nash equilibrium $\mathbf{e}$ in $\mathbf{g}$. Since all equilibria are specialised, under $\mathbf{e}$ agents will either be specialists or non-specialists. Let $S(\mathbf{e})$ be the set of specialists in $\mathbf{e}$; therefore $N - S(\mathbf{e})$ is the set of non-specialists. Choose $\epsilon \leq min\{\bar{e}, e^* - \bar{e}\}$. Consider any perturbation γ such that $|\gamma_i| < \epsilon$ and $\gamma_i + e_i \geq 0$ for all i. To determine $\mathbf{e}^{(1)}$, let us first consider a non-specialist $i \notin S(\mathbf{e})$. Since in equilibrium i must be connected to at least one specialist, $|N_i^1(\mathbf{g}) \cap S(\mathbf{e})| \geq 1$. Let $j \in N_i^1(\mathbf{g}) \cap S(\mathbf{e})$. Therefore, $e_j^{(0)} = e^* + \gamma_j$. Since $|\gamma_j| < \epsilon$, we have $\bar{e} < e_j^{(0)} < e^*$ which implies that $e_i^{(1)} = 0$. Next consider a specialist $i \in S(\mathbf{e})$. Since i is connected only to non-specialists, $(\forall j \in N_i^1(\mathbf{g}))(e_j^{(0)} = \gamma_j)$. This together with the fact that $|\gamma_j| < \epsilon$ implies that $(\forall j \in N_i^1(\mathbf{g}))(0 < e_j^{(0)} < \bar{e})$. Hence from (1), $e_i^{(1)} = e^*$. Thus we have shown that even after the perturbation all the agents revert back to their respective equilibrium effort level they deviated from, i.e., $BR(\mathbf{e} + \gamma) = \mathbf{e}$ and hence $(\forall k \geq 1)(BR^{(k)}(\mathbf{e} + \gamma) = \mathbf{e})$. Thus, all equilibria are stable. □

Example 2. *Stable equilibria: Consider the star network with four agents illustrated in figure 3. Assuming* $e^* = 1$ *there can be only two equilibria: (i) only the agent at the centre is a specialist and the rest take no effort and (ii) each peripheral agent takes 1 unit of effort whereas the core agent takes no effort at all. Both these equilibria are stable.*

3.2. **Welfare.** In the present setting we note that given a non-empty graph, not all agents will exert positive level of effort in an efficient effort profile. This is so because in case more than one agent exert positive effort in a neighbourhood those taking lower level of effort will only incur the cost of exerting effort without getting any additional benefit from it. In fact we observe from our first proposition that the same holds true at any equilibrium, that is for any non-empty network not all agents exert effort at an equilibrium. However, in spite of this no equilibrium effort profile is efficient because equilibrium is reached when the agents compare the personal benefit they derive from an effort profile with the cost they incur from exerting effort themselves whereas efficiency takes into account the aggregate benefits and costs derived from an effort profile. Thus a specialist who is not isolated in an equilibrium can improve upon overall welfare by increasing her effort to a level where the marginal benefit that accrues to herself as well as her neighbours equals its cost.

Therefore we look for those equilibria of a network which yield the highest welfare among all equilibria. Given the benefit function introduced in this paper the aggregate benefit that the agents in a network obtain in an equilibrium always equals $nb(e^*)$; there is no possibility of an agent earning benefits over and above $b(e^*)$. To see why, recall from **??** that in equilibrium every specialist is linked only to non-specialists but a non-specialist may be linked to more than one specialist. This implies that the benefit that a specialist derive in equilibrium is equal to $b(e^*)$, that is from her own effort, and even though a non-specialist i may have more than one specialist as her neighbour, the benefit she derives equals $b(e^*)$.

Thus given a network, improvement in welfare can only be brought about by reduction in the aggregate cost of efforts at the equilibrium, which leads us to the following proposition:

Proposition 3. *Consider a network* $\mathbf{g}$ *and let* $\mathbf{e^1}$ *and* $\mathbf{e^2}$ *be two equilibria on* $\mathbf{g}$*. Of the equilibria the one with fewer specialists gives rise to higher welfare and vice versa, i.e,* $W(\mathbf{e^1},\mathbf{g}) \geq W(\mathbf{e^2},\mathbf{g})$ *iff* $|S(\mathbf{e^1})| \leq |S(\mathbf{e^2})|$.

Proof. The proof is straightforward. Since at all equilibrium every agent gets benefit of the order $b(e^*)$, $\sum\limits_{i\in N} b(e^1_{i_{max}}) = \sum\limits_{i\in N} b(e^2_{i_{max}}) = nb(e^*)$. Therefore, $W(\mathbf{e^1},\mathbf{g}) \geq W(\mathbf{e^2},\mathbf{g}) \leftrightarrow \sum\limits_{i\in N} ce^1_i \leq \sum\limits_{i\in N} ce^2_i$. Recall that in equilibrium $e_i \in \{0, e^*\}$; hence $\sum\limits_{i\in N} ce^1_i \leq \sum\limits_{i\in N} ce^2_i \leftrightarrow \sum\limits_{i\in I^1} ce^*_i \leq \sum\limits_{i\in I^2} ce^*_i$ which holds if only if $|S(\mathbf{e^1})| \leq |S(\mathbf{e^2})|$. □

Hence if specialists in an equilibrium effort profile are so located that they are directly accessible by many agents in the network then these agents gain from the benefits of her effort and hence the total effort required to be sustained at the equilibrium decreases compared to other equilibria where fewer agents can access a specialist thereby leading to a rise in welfare.

Example 3. *Comparison of welfare of Nash equilibria: Consider the same network as in Example 2. As was shown, there are just two equilibria: (i) and (ii). Comparing the two equilibria it can be seen that the number of specialists in (i) is less than in (ii) and hence the equilibrium (i) yields higher welfare than the (ii). In fact since there are only two equilibria, (i) is the second best equilibrium of the network.*

Effects of new links. In a graph if new links are formed between agents while retaining the initial links, how does the welfare compare at the equilibrium? For examining the effects of link formation we compare between the equilibria yielding highest welfare before and after formation of new links, i.e., the second-best equilibria under two situations. The two propositions in this section identify the situations under which a new link may be detrimental from the point of view of welfare.

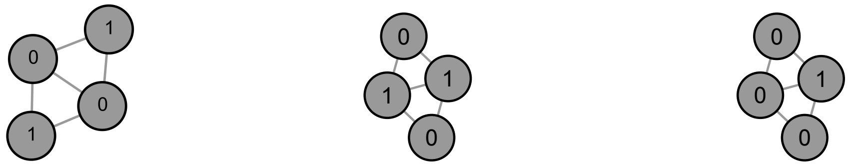
Proposition 4. *Consider two networks* $\mathbf{g}$ *and* $\mathbf{g}'$. *Let* $\mathbf{e}_{|_\mathbf{g}}$ *and* $\mathbf{e}'_{|_{\mathbf{g}'}}$ *be equilibrium of* $\mathbf{g}$ *and* $\mathbf{g}'$ *respectively. Then* $W(\mathbf{e}'_{|_{\mathbf{g}'}}; \mathbf{g}') \leq W(\mathbf{e}_{|_\mathbf{g}}; \mathbf{g})$ *iff* $|S(\mathbf{e}')_{|_{\mathbf{g}'}}| \geq |S(\mathbf{e})_{|_\mathbf{g}}|$.

The formal proof is similar to Proposition 1 and hence omitted. This result follows from the fact that if the number of individuals in two different networks is same the aggregate benefit from both the networks are equal at their respective equilibrium and hence the welfare depends only on their aggregate costs which in turn depend on the number of specialists in the equilibrium profiles.

Example 4. *Consider the figures 2(a) and 3(b) which represent equilibrium profiles for two networks. Since the number of specialists in the former is less than that in the latter, the equilibrium in the first network yields higher welfare than in the second.*

Proposition 5. *Consider two networks* $\mathbf{g}$ *and* $\mathbf{g}'(=\mathbf{g}+ij)$. *Let* $\mathbf{e}_{|_\mathbf{g}}$ *and* $\mathbf{e}'_{|_{\mathbf{g}'}}$ *be second-best equilibrium of* $\mathbf{g}$ *and* $\mathbf{g}'$ *respectively. Then* $|S(\mathbf{e}')_{|_{\mathbf{g}'}}| > |S(\mathbf{e})_{|_\mathbf{g}}|$ *only if* $e_{i_{|_\mathbf{g}}} = e^* = e_{j_{|_\mathbf{g}}}$.

Proof. Consider two networks $\mathbf{g}$ and $\mathbf{g}'$. Let $\mathbf{e}_{|_\mathbf{g}}$ and $\mathbf{e}'_{|_{\mathbf{g}'}}$ be second-best equilibrium of $\mathbf{g}$ and $\mathbf{g}'$ respectively.



(a) (b) (c)

FIGURE 4. Star

Suppose $e_{i|_\mathbf{g}} \neq e^* \vee e_{j|_\mathbf{g}} \neq e^*$. Recall that at equilibrium $e_i \in \{0, e^*\}$. Therefore $e_{i|_\mathbf{g}} = 0 \wedge e_{j|_\mathbf{g}} = e^*$ or $e_{i|_\mathbf{g}} = e^* \wedge e_{j|_\mathbf{g}} = 0$ or $e_{i|_\mathbf{g}} = e_{j|_\mathbf{g}} = 0$.

Let us first assume $e_{i|_\mathbf{g}} = 0 \wedge e_{j|_\mathbf{g}} = e^*$. Consider the profile $\mathbf{e}_{|_{\mathbf{g}'}}$. Since $\mathbf{g}$ and $\mathbf{g}'$ differ only with respect to the link between i and j, all specialists are linked only to non-specialists and every non-specialist is linked to atleast one specialist even in $\mathbf{e}_{|_{\mathbf{g}'}}$ and hence $\mathbf{e}_{|_{\mathbf{g}'}}$ is an equilibrium in $\mathbf{g}'$. Since the same effort profile $\mathbf{e}$ is an equilibrium in both $\mathbf{g}$ and $\mathbf{g}'$, following Proposition 4 we have $W(\mathbf{e}_{|_{\mathbf{g}'}}; \mathbf{g}') = W(\mathbf{e}_{|_\mathbf{g}}; \mathbf{g})$. Therefore $\mathbf{e}'_{|_{\mathbf{g}'}}$ being the second best equilibrium of $\mathbf{g}'$ imply that $W(\mathbf{e}'_{|_{\mathbf{g}'}}; \mathbf{g}') \geq W(\mathbf{e}_{|_{\mathbf{g}'}}; \mathbf{g}')$ and hence $W(\mathbf{e}'_{|_{\mathbf{g}'}}; \mathbf{g}') \geq W(\mathbf{e}_{|_\mathbf{g}}; \mathbf{g})$. This in turn implies $|S(\mathbf{e}')_{|_{\mathbf{g}'}}| \leq |S(\mathbf{e})_{|_\mathbf{g}}|$ by Proposition 4.

Next we assume that $e_{i|_\mathbf{g}} = e_{j|_\mathbf{g}} = 0$. Here since the new link is formed between two non-specialists in $\mathbf{e}_{|_\mathbf{g}}$, $\mathbf{e}$ satisfies the equilibrium condition in $\mathbf{g}'$ as well. Hence following Proposition 4, $W(\mathbf{e}_{|_{\mathbf{g}'}}; \mathbf{g}') = W(\mathbf{e}_{|_\mathbf{g}}; \mathbf{g})$ which implies $W(\mathbf{e}'_{|_{\mathbf{g}'}}; \mathbf{g}') \geq W(\mathbf{e}_{|_\mathbf{g}}; \mathbf{g})$. Therefore, by Proposition 4 we have $|S(\mathbf{e}')_{|_{\mathbf{g}'}}| \leq |S(\mathbf{e})_{|_\mathbf{g}}|$.

The argument for the third case is analogous to the first and hence omitted. □

Propositions 4 and 5 thus clearly establish that new connections in a network may lead to a loss in welfare only if the new links are formed between specialists of the network. This is so because in no other circumstances there is any possibility of the number of specialists going up in the equilibrium after the link has been formed.

The following example illustrates the significance of the results of this section.

Example 5. *Consider figure 2(a) which shows the second best equilibrium of the star network. As can be seen, there only two agents are specialists and others free ride. If a link is formed between the free riders resulting in a network as shown in figure 4(a), the second best equilibrium of the previous network remains so even in the new network. However, when the link is formed between the specialists as in figure 4(b), the equilibrium* *from the previous network ceases to be so and under the new second best equilibrium for this network, shown in figure 4(c), the number of specialists is 1 as compared to 2 in the initial network. This example demonstrates that Proposition 5 gives only a necessary condition for the reduction in welfare due to link formation and not a sufficient condition.*

4. Robustness

In this section we examine whether predictions about the equilibria in our original game are retained if we generalise our model in various directions. In doing so, we also identify which assumptions of our model are crucial in driving the results of the previous section.

4.1. **Differentiated Efforts.** In this section we consider the case where, unlike in the original game, the same level of effort may have differential impacts on the agents depending on whether it is her own effort or her neighbour's. We assume here that the benefits of the entire effort of an agent's neighbour does not reach her, in fact it is discounted by the factor δ. The benefit function is given by $b(e^d_{i_{max}})$ where $e^d_{i_{max}} = max\{\delta e_j | j \in N^1_i(\mathbf{g})\} \cup \{e_i\}$. It is to be noted that $\delta = 1$ corresponds to our basic model. Our next proposition shows that for low enough δ, there is a unique equilibrium in which every agent is a specialist whereas for higher values of δ we can apply Proposition 1.

Proposition 6. *Suppose an agent's benefit function is given by* $b(e^d_{i_{max}}), b' > 0, b'' < 0, b(0) = 0, b'(0) > c$. *where* $0 < \delta \leq 1$. *Then (i) if* $\delta < \overline{e}/e^*$*, there is a unique equilibrium in which every agent is a specialist, (ii) if* $\delta = \overline{e}/e^*$*, an effort profile* $\mathbf{e}$ *is a Nash Equilibrium in* $\mathbf{g}$ *iff it is specialised, and for every non-specialist* i, $N^1_i(\mathbf{g}) \cap S(\mathbf{e}) \neq \phi$ *and (iii) if* $\delta > \overline{e}/e^*$*, an effort profile* $\mathbf{e}$ *is a Nash Equilibrium in* $\mathbf{g}$ *iff it is specialised, and it is a maximal independent set of the network* $\mathbf{g}$.

Proof. In this case the best response function of an agent i is given by

$$BR_i(\mathbf{e}_{\sim i}; \mathbf{g}) = \begin{cases} \{e^*\} & \text{if} \quad (\forall j \in N^1_i(\mathbf{g}))(e_j < \overline{e}/\delta) \\ \{0, e^*\} & \text{if} \quad (\forall j \in N^1_i(\mathbf{g}))(e_j \leq \overline{e}/\delta) \wedge (\exists j \in N^1_i(\mathbf{g}))(e_j = \overline{e}/\delta) \\ \{0\} & \text{if} \quad (\exists j \in N^1_i(\mathbf{g}))(e_j > \overline{e}/\delta) \end{cases} \tag{4.1}$$

First consider the case $\delta < \overline{e}/e^*$. It is clear from 4.1 that as long as the maximum effort exerted by the neighbours of any agent i is less than $\overline{e}/\delta$, best response of i is to exert e^*. This, together with the facts that $\overline{e}/\delta > e^*$ and that in equilibrium no individual will exert efforts more than e^* implies that every agent exerts e^* level of effort in an equilibrium.
Next suppose $\delta = \overline{e}/e^*$. First we prove the necessity. Suppose $\mathbf{e}$ be a Nash equilibrium. That $\mathbf{e}$ is a specialised profile is clear from 4.1 as in Nash equilibrium all the agents play their best responses. Suppose there exists a non-specialist i such that $N_i^1(\mathbf{g}) \cap S(\mathbf{e}) = \phi$. Then from the best response function 4.1, it is clear that agent i is not playing her response and hence $\mathbf{e}$ cannot be a Nash equilibrium, a contradiction. This completes the necessity part of statement (ii). Now for the sufficiency part, suppose $\mathbf{e}$ is an effort profile such that it is specialised, and for every non-specialist i, $N_i^1(\mathbf{g}) \cap S(\mathbf{e}) \neq \phi$. Consider an $i \in S(\mathbf{e})$. Then the maximum that i can get from her neighbours is δe^* in which case she is playing her best response. Now take any $j \in N - S(\mathbf{e})$. Since j is linked to atleast one specialist, she is playing a best response by exerting no effort. Thus, $\mathbf{e}$ is a Nash Equilibrium. This completes the proof of statement (ii).
Finally suppose $\delta > \overline{e}/e^*$. It is easy to see that in this case the proof of Proposition 1 goes through with the best response function 3.2 replaced by 4.1. □

Remark 1. *When $\delta = \overline{e}/e^*$, there does not exist any strict Nash equilibrium as in all possible equilibria given the effort levels of their neighbours atleast the non-specialists are indifferent between choosing either e^* and 0.*

From the above result we can infer that given a network, for low values of δ welfare under the equilibrium profile is lower as compared to the second best equilibria of our original game while it remains the same in both cases for high enough δ. Moreover, formation of new links when δ is very low does not lead to a change in welfare with the number of specialists remaining the same in both cases while it may be beneficial for high enough values of δ.

4.2. **Convex costs.** In our original model we considered the linear cost of efforts for every agent. However, even if we allow for increasing and convex costs $c(e_i)$, the following proposition shows that the equilibrium condition remains the same. But in contrast to the original game, now the optimal effort level e^* of an isolated individual is the level of effort at which $b'(e) = c'(e)$. Here again the restrictions on $b(.)$ and $c(.)$ guarantee the existence of an $\overline{e}$ such that

$$\begin{aligned} b(e^*) - c(e^*) &> b(e) \quad && \text{for} \quad && e < \overline{e} \\ &< b(e) && \text{for} && e > \overline{e} \\ &= b(e) && \text{for} && e = \overline{e} \end{aligned} \tag{4.2}$$

so that the best response function of an agent i is still given by 3.2.

Proposition 7. *Suppose that $c(e_i)$ is increasing and convex and that $c'(0) > b'(\infty)$. Then an effort profile* **e** *is a Nash Equilibrium in* **g** *iff it is specialised and its set of specialists is a maximal independent set of the network* **g**.

The proof of Proposition 1 goes through although now with the $\overline{e}$ being defined according to 4.2.

The rest of the analysis of our original game still remains valid in the convex cost case and hence our results are robust to the changes in cost structure to a more general one.

4.3. **Heterogeneous agents.** In this section we assume that the agents are heterogeneous in terms of both their benefit functions as well as the cost of efforts. Let the benefit function and the cost function of an agent $i \in N$ be given by $b_i(e_{i_{max}})$ and $c_i e_i$ respectively where both b_is and c_is may be different for different agents. The case where they are same for all agents corresponds to our original game. In this case the optimal effort level of different agents when they are not linked to anyone in the network are different and are given by e_i^* with e_i^* being higher for agents who while having the same marginal costs of efforts as others derive greater benefits from the public good than others or incur lower costs of effort compared to others while having identical benefit function as others. When both the marginal costs of efforts and benefit functions are different for every individual, comparison of e_i^* would depend on both benefit and cost functions of each individual.

The best response function of an agent i, in presence of heterogeneity, is given by

$$BR_i(\mathbf{e}_{\sim i};\mathbf{g}) = \begin{cases} \{e_i^*\} & \text{if} \quad (\forall j \in N_i^1(\mathbf{g}))(e_j < \overline{e}_i) \\ \{0, e_i^*\} & \text{if} \quad (\forall j \in N_i^1(\mathbf{g}))(e_j \leq \overline{e}_i) \wedge (\exists j \in N_i^1(\mathbf{g}))(e_j = \overline{e}_i) \\ \{0\} & \text{if} \quad (\exists j \in N_i^1(\mathbf{g}))(e_j > \overline{e}_i) \end{cases} \tag{4.3}$$

where existence of $\overline{e}_i$ is guaranteed for each agent by $b_i'(0) > c_i, b' > 0, b'' < 0$. The following proposition gives a characterization of the Nash equilibrium with heterogeneous agents.

Proposition 8. *Let the benefit function and the cost function of an agent $i \in N$ be given by $b_i(e_{i_{max}})$ and $c_i e_i$ respectively. An effort profile $\mathbf{e}$ is an equilibrium for $\mathbf{g}$ iff (i) $\mathbf{e}$ is specialised (ii) $S(\mathbf{e})$ is a maximal independent set and (iii) $\forall i \in N - S(\mathbf{e}), \exists j \in N_i^1(\mathbf{g}) \cap S(\mathbf{e})$ such that $e_j^* \geq \overline{e}_i$.*

Proof. Take a Nash equilibrium profile $\mathbf{e}$. That $\mathbf{e}$ is specialised is immediate from 4.3 as every player plays her best response in equilibrium.

Next we prove that $S(\mathbf{e})$ is a maximal independent set. Suppose that $S(\mathbf{e})$ is not a maximal independent set. This implies that $\exists i, j \in S(\mathbf{e})$ such that $j \in N_i^1(\mathbf{g})$ or $\exists i \in N - S(\mathbf{e})$ such that $N_i^1(\mathbf{g}) \cap S(\mathbf{e}) = \phi$. First suppose $\exists i, j \in S(\mathbf{e})$ such that $j \in N_i^1(\mathbf{g})$. If $\overline{e}_i \geq e_j^*$, then it implies that $e_i^* > \overline{e}_j$ which by 4.3 implies that agent j is not playing her best response whereas if $\overline{e}_i < e_j^*$, then agent i is not playing her best response and can improve her payoff by switching to exerting no effort which contradicts our hypothesis that $\mathbf{e}$ is a Nash equilibrium. Similar arguments hold when $\overline{e}_j <$ or $\geq e_i^*$. Next suppose $\exists i \in N - S(\mathbf{e})$ such that $N_i^1(\mathbf{g}) \cap S(\mathbf{e}) = \phi$. But then i can profitably switch to exerting e_i^* following 4.3 which contradicts our assumption that $\mathbf{e}$ is a Nash equilibrium. Hence (ii) must hold.

Lastly, suppose $\exists i \in N - S(\mathbf{e})$ such that $\forall j \in N_i^1(\mathbf{g}) \cap S(\mathbf{e}), e_j^* < \overline{e}_i$. This, following 4.3, implies i can improve her payoff by exerting e_i^* level of effort as she is not playing her best response. This is a contradiction to our assumption that $\mathbf{e}$ is a Nash equilibrium. Hence (iii) must hold.

Consider an effort profile $\mathbf{e}$ which satisfies (i), (ii) and (iii). Consider an $i \in N - S(\mathbf{e})$.

Since she is connected to atleast one specialist j such that $e_j^* \geq \overline{e}_i$, we can infer from 4.3 that she plays her best response by not exerting any effort. Next consider any $i \in S(\mathbf{e})$. Since $S(\mathbf{e})$ is a maximal independent set, $\forall j \in N_i^1(\mathbf{g}), e_j = 0$. Hence by 4.3 she plays her best response by exerting e_i^* level of effort. Therefore, $\mathbf{e}$ is a Nash equilibrium. □

Remark 2. *An equilibrium profile as also any specialised profile in the heterogeneous agents case, though involve the agents either taking no effort or the effort they would have taken had they been isolated, consists of various different effort levels apart from zero because of the fact that different individuals have different e_i^*s.*

In this case we note that in general, for any network there may be fewer Nash equilibria than in the original model. Moreover, welfare under different equilibria in a network or across networks no longer varies only according to the number of specialists in those profiles but it depends both on the cost and benefit structure of each agent.

4.4. **Beyond pure local spillovers.** Situations where agents can benefit from the efforts of agents who are not immediate neighbours but are distant neighbours located farther away in the network are not inconceivable. For instance, benefits from innovation of a single firm may reach at least L steps without any decay. In this section we characterize the equilibrium under this modified setting where we assume that the benefits of the efforts taken by an agent can reach upto the neighbours located at a distance L from her; in other words we allow for indirect transmission of benefits upto distance L. Although Friedkin (1983) found from his study of communication networks that information diffuses not more than two steps, to make our analysis more general we allow L to take any value between 1 and $n-1$. It is to be observed that the case where $L = 1$ corresponds to our basic model of pure local spillover.

The benefit function under this setting is given by $b(e_{i_{max}}), b' > 0, b'' < 0, b(0) = 0, b'(0) > c$ where $e_{i_{max}}^{IT} = max\{e_j | j \in \bigcup_{l=1}^{L} N_i^l(\mathbf{g}) \cup \{i\}\}$; $L \in \{1, \ldots, n-1\}$.

Proposition 9. *Let the benefit function of an agent $i \in N$ be given by $b_i(e_{i_{max}}^{IT})$. An effort profile $\mathbf{e}$ is an equilibrium for $\mathbf{g}$ iff (i) $\mathbf{e}$ is specialised, and (ii) $\forall i, j \in S(\mathbf{e}), d(i, j; \mathbf{g}) > L$ and $\forall j \in N - S(\mathbf{e}), \exists i \in S(\mathbf{e})$ such that $d(i, j; \mathbf{g}) \leq L$.*

Proof. The best response function of an agent i under indirect transmission of benefits is given by

$$BR_i(\mathbf{e}_{\sim i};\mathbf{g}) = \begin{cases} \{e^*\} & \text{if} \quad (\forall j \in \bigcup_{l=1}^{L} N_i^l(\mathbf{g}))(e_j < \overline{e}) \\ \{0, e^*\} & \text{if} \quad (\forall j \in \bigcup_{l=1}^{L} N_i^l(\mathbf{g}))(e_j \leq \overline{e}) \wedge (\exists j \in \bigcup_{l=1}^{L} N_i^1(\mathbf{g}))(e_j = \overline{e}) \\ \{0\} & \text{if} \quad (\exists j \in \bigcup_{l=1}^{L} N_i^l(\mathbf{g}))(e_j > \overline{e}) \end{cases} \tag{4.4}$$

First we prove the necessity part. Suppose $\mathbf{e}$ is an equilibrium. The necessity of part (i) follows immediately from 4.4 since at the equilibrium all agents play their best responses. To prove necessity of condition, first suppose $\exists i, j \in S(\mathbf{e})$ such that $d(i, j; \mathbf{g}) \leq L$. Then according to 4.4, agent i is not playing her best response which is in violation of the fact that $\mathbf{e}$ is an equilibrium. Next suppose $\exists j \in N - S(\mathbf{e})$ such that $\forall i \in S(\mathbf{e}), d(i, j; \mathbf{g}) > L$. This implies that $(\forall j' \in \bigcup_{l=1}^{L} N_j^l(\mathbf{g}))(e'_j = 0)$. Then, clearly by 4.4, agent j is not playing her best response, a contradiction. This completes the necessity proof.

Now we move on to prove the sufficiency part. Take any effort profile $\mathbf{e}$ such that conditions (i)- (ii) are satisfied. Consider any $i \in S(\mathbf{e})$. Since $\forall j \in \bigcup_{l=1}^{L} N_i^l(\mathbf{g}))(e_j = 0)$, i is clearly playing her best response, as is evident from 4.4. Next consider an agent $j \in N - S(\mathbf{e})$. Since $(\exists j' \in \bigcup_{l=1}^{L} N_{j'}^l(\mathbf{g}))(e_{j'} = e^*)$, agent j cannot improve her payoff by exerting any other effort level, as can be seen from 4.4. Thus $\mathbf{e}$ is a Nash equilibrium. This concludes the proof. □

Remark 3. *When L=1, condition (ii) boils down to the set of specialists forming a maximal independent set of the graph* $\mathbf{g}$ *which together with condition (i) yield the same characterising conditions as we have obtained in Proposition 1.*

The other way of characterizing Nash equilibria is by constructing a new graph $\mathbf{g}^L$ from the graph $\mathbf{g}$ such that $g_{ij}^L = 1$ if $j \in \bigcup_{l=1}^{L} N_i^l(\mathbf{g}))$ and 0 otherwise[7]. The spillover to immediate neighbours in the new graph $\mathbf{g}^L$, is equivalent to the spillover upto a distance L in the graph $\mathbf{g}$ and hence the the result in the proposition 1 holds for $\mathbf{g}^L$. Hence equilibrium

[7]This construction is due to Bramoulle and Kranton (2005), which is essentially a longer version of Bramoulle and Kranton (2007).

in $\mathbf{g}$ can be characterized by using the properties of the graph $\mathbf{g}^L$. Though this is a straightforward extension of a result obtained earlier in the paper, in our opinion it is much less appealing than the characterization in terms of the properties of the original network.

5. **Conclusion**

In this paper we analyze a setting of information sharing which is known as Best Shot Network Game (BSNG) but unlike in the literature we consider the strategy space to be continuous. Though this setting is very simple it helps us understand incentive structure in many problems of local contribution as discussed above and much of the results from the two-action space, as discussed in Jackson (2008), Boncinelli and Pin (2012), Galeotti et al. (2010) carry over to our model, though there are some important differences. We have shown that irrespective of the structure of the network the pattern of public good provision will be invariably such that some agents will contribute while others will free ride. This is the result obtained when distributed profiles exist under our set up whereas by construction the distributed profiles do not exist in BSNG with action space $\{0, 1\}$ where agents only choose between exerting effort or not and do not have the option to choose effort level. It may be noted that the reduction in the number of equilibrium as compared to Bramoulle and Kranton (2007) hinges entirely on our benefit function. However, Nash tatonnement fails to have any selecting power in our setting since all equilibria turn out to be stable which in fact expands the set of stable equilibrium when compared to Bramoulle and Kranton (2007). Upon welfare evaluation our model suggests that given a network, the lower the number of contributors the better it is for the society and integration of various groups through formation of ties between them may at times be counter-productive. Though in BSNG with two-action space the equilibrium profiles with minimum number of specialists turn out to be efficient, in our model no equilibrium is efficient, thereby reinforcing the insights from the standard game of private provision of public good. Since our model could not overcome the problem of multiplicity of equilibrium, it would be interesting to explore whether other notions of stability like asymptotic stability (Bramoulle et al. (2014)) and stochastic stability (Boncinelli and

Pin (2012)) further refines the set of equilibria and improves the predictive power of the model.

References

Ballester, C., Calvó-Armengol, A., and Zenou, Y. (2006). Who's who in networks. wanted: The key player. *Econometrica*, 74(5):1403–1417.

Boncinelli, L. and Pin, P. (2012). Stochastic stability in best shot network games. *Games and Economic Behaviour*, 75:538–554.

Bramoulle, Y. and Kranton, R. (2005). Strategic experimentation in networks. NajEcon Working Paper Reviews 784828000000000417, www.najecon.org.

Bramoulle, Y. and Kranton, R. (2007). Public goods in networks. *Journal of Economic Theory*, 135:478–494.

Bramoulle, Y., Kranton, R., and D'Amours, M. (2014). Strategic Interaction and networks. *American Economic Review*, 104(3):898–930.

Corbo, J., Calvó-Armengol, A., and Parkes, D. C. (2007). The importance of network topology in local contribution games. In Deng, X. and Graham, F. C., editors, *Internet and Network Economics: Third International Workshop, WINE 2007, San Diego, CA, USA, December 12-14, 2007. Proceedings*, pages 388–395. Springer Berlin Heidelberg.

Friedkin, N. E. (1983). Horizons of observability and limits of informal control in organizations. *Social forces*, 62(1):54–77.

Fudenberg, D. and Tirole, J. (1991). *Game Theory.* The MIT Press, Cambridge, Massachusetts.

Galeotti, A., Goyal, S., Jackson, M. O., Vega-Redondo, F., and Yariv, L. (2010). Network games. *Review of Economic Studies*, 77:218–244.

Goyal, S. (2007). *Connections, An Introduction to the economics of Networks.* Princeton University Press.

Harrison, G. W. and Hirshleifer, J. (1989). An experimental evaluation of weakest-link/ best-shot models of public goods. *Journal of Political Economy*, 97(1):201–225.

Hirshleifer, J. (1983). From weakest-link to best-shot: The voluntary provision of public goods. *Public Choice*, 41(3):371–386.

Jackson, M. O. (2008). *Social and Economic Networks*. Princeton University Press.

Jackson, M. O. and Zenou, Y. (2014). Games on networks. In Young, P. and Zamir, S., editors, *Handbook of Game Theory*, volume 4. Elsevier Science, Amsterdam.

López-Pintado, D. (2013). Public goods in directed networks. *Economics Letters*, 121(2):160–162.